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Using the notation from the article by T. Bajbar and O. Stein, "Coercive polynomials and their Newton polytopes"[1], the aim of this note is to prove that there are non-coercive polynomials that are coercive on the curves $x_{y,\beta}(t) = (y_1 e^{\beta_1 t}, \dots, y_n e^{\beta_n t})$ with $(y, \beta) \in \Omega$:

Proposition 1. *Let $f \in \mathbb{R}[X_1, X_2]$ given by*

$$f = (X_2 - X_1 - 1)^2 (X_1^2 + X_2^2).$$

Then f is not coercive but $\Omega \subset \Omega_f$.

The notion of an *asymptotic direction* of a curve captures some of the behaviour of the curve at infinity. Asymptotic directions are elements of the $(n-1)$-sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Definition 2. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be continuous with $\lim_{t\to+\infty} \|\gamma(t)\| = +\infty$. We say the curve γ has the asymptotic direction $\omega \in \mathbb{S}^{n-1}$ if

$$\lim_{t\to+\infty} \frac{\gamma(t)}{\|\gamma(t)\|} = \omega,$$

or $\mathcal{D}(\gamma) = \omega$ for short.

We can now compute the asymptotic directions of the curves $x_{y,\beta}$ in $n = 2$ dimensions. To this end let $\operatorname{sgn}(y)$ be the sign of $y \in \mathbb{R}$ where $\operatorname{sgn}(0) := 0$.

Lemma 3. *Let $n = 2$ and $(y, \beta) \in \Omega$. Then an asymptotic direction for $x_{y,\beta}$ exists. More precisely, for β we have exactly one of the following cases:*

a) $\beta_1 = 0$. Then $\mathcal{D}(x_{y,\beta}) = (0, \operatorname{sgn}(y_2))$ and $x_{y,\beta}$ is a line parallel to the x_2-axis.

b) $\beta_2 = 0$. Then $\mathcal{D}(x_{y,\beta}) = (\operatorname{sgn}(y_1), 0)$ and $x_{y,\beta}$ is a line parallel to the x_1-axis.

c) $\beta_1 < 0$. Then $\mathcal{D}(x_{y,\beta}) = (0, \operatorname{sgn}(y_2))$.

d) $\beta_2 < 0$. Then $\mathcal{D}(x_{y,\beta}) = (\operatorname{sgn}(y_1), 0)$.

e) $\beta_1 = \beta_2 > 0$. Then $\mathcal{D}(x_{y,\beta}) = \frac{y}{\|y\|}$ and $x_{y,\beta}$ is a line through the origin.

f) $\beta_1 > \beta_2 > 0$. Then $\mathcal{D}(x_{y,\beta}) = (\operatorname{sgn}(y_1), 0)$.

g) $\beta_2 > \beta_1 > 0$. Then $\mathcal{D}(x_{y,\beta}) = (0, \operatorname{sgn}(y_2))$.

[1] Tomáš Bajbar and Oliver Stein. Coercive polynomials and their Newton polytopes. *SIAM Journal on Optimization*, 25(3):1542–1570, 2015.

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Proof. Note that by definition of Y, $y_i \neq 0$ for all $i \in [n]$ throughout this proof. As $\beta \in B$, we have $\beta_1 \leq 0 \Rightarrow \beta_2 > 0$ and $\beta_2 \leq 0 \Rightarrow \beta_1 > 0$. Hence, a), b), c) and d) follow by standard arguments, we show c) as an example. Indeed, the first component of $x_{y,\beta}(t)/\|x_{y,\beta}(t)\|$ converges to zero for $t \to +\infty$ as the nominator is bounded and the denominator is unbounded. The second component is

$$\frac{y_2 e^{\beta_2 t}}{\sqrt{y_1^2 e^{2\beta_1 t} + y_2^2 e^{2\beta_2 t}}} = \frac{y_2 e^{\beta_2 t}}{|y_2| e^{\beta_2 t}} \frac{1}{\sqrt{\frac{y_1^2}{y_2^2} e^{2(\beta_1 - \beta_2)t} + 1}} = \frac{\operatorname{sgn}(y_2)}{\sqrt{\frac{y_1^2}{y_2^2} e^{2(\beta_1 - \beta_2)t} + 1}},$$

and the denominator converges to 1 for $t \to +\infty$, as $\beta_1 < 0$ and $\beta_2 > 0$.

Also, e) is clear. To see f), we observe

$$(y_1 e^{\beta_1 t}, y_2 e^{\beta_2 t}) = e^{\beta_2 t}(y_1 e^{(\beta_1 - \beta_2)t}, y_2),$$

and by absolute homogeneity of the norm, the factor $e^{\beta_2 t}$ has no influence on the asymptotic direction. So we may neglect it, and the asymptotic direction is the same as in b), i.e. $\mathcal{D}(x_{y,(\beta_1,\beta_2)}) = \mathcal{D}(x_{y,(\beta_1-\beta_2,0)}) = (\operatorname{sgn}(y_1), 0)$. The proof for g) is similar. □

The following lemma allows us to prove Proposition 1. Roughly speaking, the lemma says that if the asymptotic direction of a curve γ exists and is not parallel to one of the two "asymptotic directions" of the zero locus of $g(x_1, x_2) := (x_2 - x_1 - 1)^2$, g cannot get arbitrarily small on $\gamma(t)$ for t large.

Lemma 4. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be continuous with $\lim_{t\to+\infty} \|\gamma(t)\| = +\infty$ and $\mathcal{D}(\gamma) = \omega \in \mathbb{S}^1$. Put $g := (X_2 - X_1 - 1)^2$. If $\omega \neq \pm\frac{(1,1)}{\|(1,1)\|}$ there is $t_0 \in \mathbb{R}$ with*

$$g(\gamma(t)) \geq 1, \quad t \geq t_0.$$

Proof. Suppose the contrary and let $\gamma = (\gamma_1, \gamma_2)$. Thus, for every $n \in \mathbb{N}$ there is $t_n \geq n$ with

$$(\gamma_2(t_n) - \gamma_1(t_n) - 1)^2 < 1. \tag{1}$$

Now we add zero to find

$$\frac{(\gamma_1(t_n), \gamma_2(t_n))}{\|\gamma(t_n)\|} = \frac{(\gamma_1(t_n), \gamma_2(t_n) - \gamma_1(t_n) - 1 + \gamma_1(t_n) + 1)}{\|\gamma(t_n)\|}.$$

We observe, using equation (1) and $\gamma(t) \to \infty$ for $t \to \infty$, that

$$\lim_{n\to\infty} \underbrace{\frac{\gamma_2(t_n) - \gamma_1(t_n) - 1}{\|\gamma(t_n)\|}}_{=:A_n} \to 0, \quad \lim_{n\to\infty} \underbrace{\frac{1}{\|\gamma(t_n)\|}}_{=:B_n} = 0.$$

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This implies

$$\begin{aligned}\omega &= \lim_{n\to\infty} \frac{\gamma(t_n)}{\|\gamma(t_n)\|} = \lim_{n\to\infty} \left(\frac{\gamma_1(t_n)}{\|\gamma(t_n)\|}, \frac{\gamma_1(t_n)}{\|\gamma(t_n)\|} + A_n + B_n \right) \\ &= \lim_{n\to\infty} \frac{(\gamma_1(t_n), \gamma_1(t_n))}{\|\gamma(t_n)\|} = (\omega_1, \omega_1),\end{aligned}$$

hence $\omega_2 = \omega_1$. However, as $\omega \in \mathbb{S}^1$, this forces $\omega = \pm(1,1)/\|(1,1)\|$, contradicting the assumption on ω. □

Proof of Proposition 1. To see that $f(x_1, x_2) = (x_2 - x_1 - 1)^2(x_1^2 + x_2^2)$, is not coercive, we observe that $f = 0$ on the line $x_2 = x_1 + 1$. To prove that $\Omega \subset \Omega_f$ we need to show

$$\lim_{t\to+\infty} \pi_f(y, \beta, t) = \lim_{t\to+\infty} f(x_{y,\beta}(t)) = +\infty, \quad (y, \beta) \in \Omega.$$

It is enough to show that $(x_2 - x_1 - 1)^2 \geq 1$ on $x_{y,\beta}(t)$ for large t, as the term $x_1^2 + x_2^2$ grows without bound for large t on the curve $x_{y,\beta}(t)$. We make the same case distinction on all possible values of β. In view of Lemmata 3 and 4, it is now clear that all choices of β except possibly $\beta_1 = \beta_2 > 0$, that is case e), imply coercive behaviour of f on $x_{y,\beta}(t)$. So let $\beta_1 = \beta_2 > 0$, hence $x_{y,\beta}$ suffices $\mathcal{D}(x_{y,\beta}) = y/\|y\|$. Lemma 4 tells us that we only need to consider the cases $y/\|y\| = \pm(1,1)/\|(1,1)\|$. However, we also know that $x_{y,\beta}$ is a line through the origin, more precisely of the form $(x_{y,\beta})_2(t) = (x_{y,\beta})_1(t)$ – hence $|(x_{y,\beta})_2(t) - (x_{y,\beta})_1(t) - 1| = 1$ for all t, and f is coercive along $x_{y,\beta}(t)$ in this case as well. □