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A note on the dual description of projected polytopes

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Abstract

The inequalities which describe the projection Q of a given polytope P onto a subspace are usually generated by an elimination procedure of Fourier-Motzkin type. In this note we give a dual approach for the description of Q. In fact, the vertices of a dual polytope serve as indices for the describing inequalities. Moreover we show how the redundancy of inequalities is connected with the existence of Slater points in the images of a set-valued mapping.

Key words: Polytope, linear programming, duality

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1. Introduction. For $m,\ n,\ p \in I\!N$ consider an arbitrary function $a : I\!R^n \to I\!R^p$, a $(p \times m)-$matrix B and the set

$$P \;=\; \{\; (x,y) \in I\!R^n \times I\!R^m |\; a(x) + By \;\geq\; 0 \;\} \;,$$

where the inequality is to be understood componentwise. We are interested in a description of the orthogonal projection $\pi(P)$ of P onto the first n variables, i.e. $Q = \pi(P) \subset I\!R^n$. In contrast to the standard procedure of Fourier-Motzkin type elimination (cf., e.g., [5],[7] and the references therein) we give a dual description of Q in this note. Unlike the dual approach in [3], our description of Q is explicit up to the determination of certain vertices of a dual polytope, as it is shown in Theorems 3 and 7. Our use of set-valued mappings enables us to give a sufficient condition for redundancy of certain vertices in terms of the Slater condition. Moreover, our approach carries over to the case where P is a polytope and π is some arbitrary projection operator, which we study in Theorem 9.

2. The dual description. Since P is the graph of the set-valued mapping

$$\Gamma :\; I\!R^n \longrightarrow 2^{I\!R^m}, \quad x \longmapsto \{\, y \in I\!R^m |\; a(x) + By \;\geq\; 0 \,\} \;,$$

we have $x \in Q$ if and only if $\Gamma(x) \neq \emptyset$. Note that the images of Γ are polyhedra. The following assumption is supposed to hold throughout this note:

Assumption 1 *The set-valued mapping Γ has bounded images.*

For fixed x we now consider the following (primal) optimization problem:

$$P(x) : \qquad \max_{(y,z)} z \qquad \text{s.t.} \qquad a(x) + By \;\geq\; z \cdot e \;,$$

where $e = (1, ..., 1)^\top \in I\!R^p$, and z is a scalar. Its dual problem is

$$D(x) : \qquad \min_{\mu} \mu^\top a(x) \qquad \text{s.t.} \qquad B^\top \mu = 0,\; e^\top \mu = 1,\; \mu \geq 0 \;.$$

Now let

$$Z_P(x) \;=\; \{\; (y,z) \in I\!R^m \times I\!R^1 |\; a(x) + By \geq ze \;\}$$

and

$$Z_D \;=\; \{\; \mu \in I\!R^p |\; B^\top \mu = 0,\; e^\top \mu = 1,\; \mu \geq 0 \;\}$$

denote the feasible sets of $P(x)$ and $D(x)$, respectively. Observe that Z_D neither depends on x nor on y.

Lemma 1 *The following assertions hold:*

(i) Z_D *is non-empty and bounded.*

(ii) For each $\bar{x} \in \mathbb{R}^n$*, both* $P(\bar{x})$ *and* $D(\bar{x})$ *are solvable.*

Proof. In assertion (i), Z_D is bounded as a subset of the standard simplex. Assume that Z_D is empty. Then, by the duality theorem of linear programming, for given $\bar{x} \in \mathbb{R}^n$ either $Z_P(\bar{x})$ is empty or the objective function z of $P(\bar{x})$ is not bounded from above on $Z_P(\bar{x})$. As the set $Z_P(\bar{x})$ always contains the point $(y, z) = (0, \min_{1 \le i \le p} a_i(\bar{x}))$, it is non-empty. Consequently, there is a sequence (y^ν, z^ν) with

$$a(\bar{x}) + By^\nu \ \ge \ z^\nu \, e \tag{1}$$

and $z^\nu \to +\infty$. For some $\nu_0 \in \mathbb{N}$ and all $\nu \ge \nu_0$ we have $y^\nu \in \Gamma(\bar{x})$ so that, without loss of generality, $y^\nu \to \bar{y} \in \Gamma(\bar{x})$ by the compactness of $\Gamma(\bar{x})$. However, then (1) cannot hold for arbitrarily large $\nu \in \mathbb{N}$. Contradiction. Assertion (ii) follows from (i) in virtue of the duality theorem. •

Let V denote the vertex set of Z_D, i.e. the set of points in Z_D where p of the active constraints are linearly independent.

Lemma 2 *For each* $\bar{x} \in \mathbb{R}^n$ *the following are equivalent:*

(i) $\Gamma(\bar{x})$ *is non-empty.*

(ii) $\min_{\mu \in V} \ \mu^\top a(\bar{x}) \ \ge \ 0$.

Proof. The set $\Gamma(\bar{x})$ is non-empty if and only if there exists a point $(y, z) \in Z_P(\bar{x})$ with $z \ge 0$. As $P(\bar{x})$ is solvable by Lemma 1(ii), the latter holds if and only if the optimal value of $P(\bar{x})$ is non-negative. The duality theorem now implies the equivalence of (i) and

$$\min_{\mu \in Z_D} \ \mu^\top a(\bar{x}) \ \ge \ 0 \ . \tag{2}$$

By the vertex theorem of linear programming, (2) implies (ii). Since $D(\bar{x})$ is solvable by Lemma 1(ii), assertion (ii) also implies (2). •

As we have $Q = \{ \, x \in \mathbb{R}^n | \ \Gamma(x) \neq \emptyset \, \}$, the next assertion follows immediately from Lemma 2:

Theorem 3 *For the set* $Q = \pi(P)$ *the following description holds:*

$$Q \; = \; \bigcap_{\mu \in V} \{\, x \in I\!R^n |\ \mu^\top a(x) \geq 0 \,\} \; . \tag{3}$$

Theorem 3 gives a description of Q by finitely many inequalities, where V serves as the finite index set. Observe that the defining functions for the inequalities are linear combinations of the functions a_i, $1 \leq i \leq p$. In order to obtain an explicit description of Q, a vertex enumeration algorithm can be applied to the polytope Z_D (cf. [1]). A similar description of Q is given in [4], but there the function a is supposed to be affine linear. Moreover, our approach via a set-valued mapping gives rise to a sufficient criterion for redundant constraints, which we investigate in the next section.

3. Redundant constraints. Like in the Fourier-Motzkin elimination procedure, some of the inequalities corresponding to vertices $\mu \in V$ may be redundant for the description of Q. We will not dwell on a minimal representation of Q in this note (cf. [2, 6]), but we only give a sufficient condition for redundancy in this section.

Definition 4 *For given* $\bar{x} \in I\!R^n$ *we call* $\bar{y}$ *a Slater point of* $\Gamma(\bar{x})$ *if* $a(\bar{x}) + B\bar{y} > 0$.

Lemma 5 *For each* $\bar{x} \in I\!R^n$ *the following are equivalent:*

(i) $\Gamma(\bar{x})$ *possesses a Slater point.*

(ii) $\min_{\mu \in V} \mu^\top a(\bar{x}) \; > \; 0$.

Proof. The set $\Gamma(\bar{x})$ possesses a Slater point if and only if there exists a point $(y, z) \in Z_P(\bar{x})$ with $z > 0$. The equivalence of (i) and (ii) now follows with the same arguments as in the proof of Lemma 2. •

Lemmas 2 and 5 immediately yield:

Corollary 6 *For each* $\bar{x} \in I\!R^n$ *the following are equivalent:*

(i) $\Gamma(\bar{x})$ *is non-empty and does not possess a Slater point.*

(ii) $\min_{\mu \in V} \mu^\top a(\bar{x}) \; = \; 0$.

Subsequently let the following assumption hold:

Assumption 2 *The function a is lower semi-continuous.*

For given $\bar{x} \in Q$ define the active index set

$$V_0(\bar{x}) \;=\; \{\, \mu \in V \mid \mu^\top a(\bar{x}) = 0 \,\} \; .$$

If $V_0(\bar{x})$ is non-empty, then in a neighborhood of $\bar{x}$ the set Q coincides with the set

$$Q_{(\bar{x})} \;=\; \bigcap_{\mu \in V_0(\bar{x})} \{\, x \in I\!R^n \mid \mu^\top a(x) \geq 0 \,\}$$

in view of Assumption 2. In the case when $V_0(\bar{x})$ is empty, $\bar{x}$ lies in the topological interior of Q. Consequently, at least the vertices $\mu \in V$ which do not belong to

$$\tilde{V} \;=\; \bigcup_{x \in Q} V_0(x) \; ,$$

are redundant for the description of Q. To be more precise, they are strongly redundant in the sense that they are not active at any point in Q ([6]). In view of Lemma 2 and Corollary 6, a point $\bar{x} \in Q$ possesses a non-empty active index set $V_0(\bar{x})$ if and only if $\Gamma(\bar{x})$ is non-empty and does not possess a Slater point. This proves the following result.

Theorem 7 *In the characterization (3) the set V can be replaced by its subset*

$$\tilde{V} \;=\; \bigcup_{x \in \tilde{Q}} V_0(x) \; ,$$

where

$$\tilde{Q} \;=\; \{\, x \in I\!R^n \mid \Gamma(x) \text{ is non-empty and does not possess a Slater point} \,\} \; .$$

The following example shows that $\tilde{V}$ may be a proper subset of V.

Example 8
For $m = 1$, $n = 2$, $p = 3$ put $a(x) = (-x_1^2 - x_2^2, -\frac{1}{2}x_1, 1)^\top$ and $B = (1, 1, -1)^\top$, i.e. we consider

$$P \;=\; \{\; (x, y) \in I\!R^3 \mid x_1^2 + x_2^2 \leq y \leq 1,\; x_1 \leq 2y \;\} \; .$$

Assumptions 1 and 2 are clearly satisfied. After a short calculation we obtain

$$Z_D \;=\; \{\; (\lambda, \tfrac{1}{2} - \lambda, \tfrac{1}{2})^\top,\; \lambda \in [0, \tfrac{1}{2}] \;\}$$

and

$$V = \{ \tfrac{1}{2}(0,1,1)^\top, \tfrac{1}{2}(1,0,1)^\top \} ,$$

so that (3) yields

$$\begin{aligned} Q &= \{ x \in I\!R^2 |\ (0,1,1)\, a(x) \geq 0,\ (1,0,1)\, a(x) \geq 0 \} \\ &= \{ x \in I\!R^2 |\ x_1 \leq 2,\ x_1^2 + x_2^2 \leq 1 \} . \end{aligned}$$

Obviously the first inequality, corresponding to the vertex $\frac{1}{2}(0,1,1)^\top \in V \setminus \tilde{V}$*, is strongly redundant for the description of* Q*.*

4. Arbitrary projection operators. The results of Theorems 3 and 7 hold particularly in the case where the functions $a_i(x) = c_i^\top x + d_i$, $1 \leq i \leq p$, are affine-linear and P is bounded, i.e. when P is a polytope. Assumptions 1 and 2 are then clearly satisfied. Moreover, in this case our method works for any projection operator after a suitable linear change of coordinates. An endomorphism $\sigma : I\!R^N \to I\!R^N$ is called projection operator if $\sigma \circ \sigma = \sigma$. We denote by $\operatorname{Im} \sigma$ and $\operatorname{Ker} \sigma$ the image and the kernel of σ, respectively.

Theorem 9 *Let* σ *be a projection operator on* $I\!R^N$*, and let*

$$\mathcal{P} = \{ u \in I\!R^N |\ \Phi\, u + \varphi \geq 0 \}$$

be a polytope with a $(p \times N)-$*matrix* Φ *and* $\varphi \in I\!R^p$*. Then we have:*

$$\sigma(\mathcal{P}) = \bigcap_{\mu \in V} \{ \xi \in \operatorname{Im} \sigma |\ \mu^\top (\Phi\, \xi + \varphi) \geq 0 \} ,$$

where V *denotes the vertex set of the polytope*

$$Z_D = \{ \mu \in I\!R^p |\ \Phi^\top \mu \in (\operatorname{Ker} \sigma)^\perp,\ e^\top \mu = 1,\ \mu \geq 0 \} .$$

Proof. Let the columns of the $(N \times n)-$matrix T_1 and the columns of the $(N \times m)-$matrix T_2 form a basis of $\operatorname{Im} \sigma$ and $\operatorname{Ker} \sigma$, respectively. The columns of T_1 are eigenvectors of σ to the eigenvalue 1, whereas the columns of T_2 are eigenvectors to the eigenvalue 0. As $I\!R^N = \operatorname{Im} \sigma \oplus \operatorname{Ker} \sigma$, we have $N = n + m$, and the columns of the matrix $T = (T_1, T_2)$ form a basis of $I\!R^N$. Hence, σ is diagonalizable with $\sigma \circ T = T \circ \pi$. Here π possesses the matrix representation $\begin{pmatrix} E_n & 0 \\ 0 & 0_m \end{pmatrix}$, where E_n and 0_m denote the $(n \times n)-$identity and the $(m \times m)-$zero matrix, respectively.

Next, it is easily seen that $\sigma(\mathcal{P}) = T(\pi(P))$, where

$$P = \{v \in I\!R^N |\ (\Phi T)\, v + \varphi \geq 0\} .$$

P is again a polytope (cf., e.g., [7]), and we can apply Theorem 3 with $x = (v_1, ..., v_n)$, $y = (v_{n+1}, ..., v_{n+m})$, $a(x) = \Phi T_1\, x + \varphi$, and $B = \Phi T_2$, in order to determine $Q \times \{0\} = \pi(P)$. We arrive at

$$\begin{aligned} \sigma(\mathcal{P}) &= T(\pi(P)) = T_1(Q) = \bigcap_{\mu \in V} \{\ T_1 x |\ x \in I\!R^n,\ \mu^\top (\Phi T_1\, x + \varphi) \geq 0\ \} \\ &= \bigcap_{\mu \in V} \{\ \xi \in \operatorname{Im} \sigma |\ \mu^\top (\Phi \xi + \varphi) \geq 0\ \} , \end{aligned}$$

where V denotes the vertex set of

$$Z_D = \{\ \mu \in I\!R^p |\ T_2^\top \Phi^\top \mu = 0,\ e^\top \mu = 1,\ \mu \geq 0\ \} .$$

This shows the assertion. •

5. Final remarks. An algorithmic implementation of the description for Q in (3) obviously relies on an efficient vertex enumeration method for the determination of the vertex set V of the polytope Z_D. Here, the reverse search algorithm of Avis and Fukuda ([1]) can be applied, preferably in a version adapted to the special structure of Z_D.

This implementation as well as a comparision of our dual description method and Fourier-Motzkin-type methods with respect to efficiency and generation, respectively detection, of redundant constraints is beyond the scope of this note and will be subject of future research.

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References

[1] D. Avis, K. Fukuda, *A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra,* Discrete and Computational Geometry, Vol. 8 (1992), 295-313.

[2] E. Balas, *Projection with a minimal system of inequalities,* Computational Optimization and Applications, Vol. 10 (1998), 189-193.

[3] G. B. Dantzig, B. C. Eaves, *Fourier-Motzkin elimination and its dual,* J. Combin. Theory, Ser. A, Vol. 14 (1973), 288-297.

[4] G. L. Nemhauser, L. A. Wolsey, *Integer and Combinatorial Optimization,* Wiley, New York, 1988.

[5] J. Stoer, C. Witzgall, *Convexity and Optimization in Finite Dimensions I,* Springer-Verlag, Berlin, 1970.

[6] J. Telgen, *Minimal representation of convex polyhedral sets,* JOTA, Vol. 38 (1982), 1-24.

[7] G. Ziegler, *Lectures on Polytopes,* Springer-Verlag, New York, 1995.